

A FAMILY OF RELATIVISTIC BARGMANN-TYPE TRANSFORMS ATTACHED TO MAASS LAPLACIANS ON THE POINCARÉ DISK

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ABSTRACT. We construct a set of coherent states through special superpositions of eigenstates of a relativistic pseudoharmonic oscillator. In each superposition the coefficients are chosen to be L^2 -eigenfunctions of a σ -weight Maass Laplacian on the Poincaré disk, which are associated with the eigenvalue $4m(\sigma - 1 + m)$, $m = 0, 1, \dots, [(\sigma - 1)/2]$. For each integer m the obtained coherent states transform constitutes a of relativistic Bargmann-type transform whose integral kernel is expressed in terms of a special Appel-Kampé de Fériet's hypergeometric function.

1. INTRODUCTION

In [1] V. Bargmann has introduced a second transform labeled by a parameter $\delta > 0$ as

$$B_\delta : L^2 \left(\mathbb{R}_+, \frac{x^\delta}{\Gamma(1+\delta)} dx \right) \rightarrow \mathcal{A}^{\delta+1}(\mathbb{D}) \quad (1.1)$$

defined by

$$B_\delta [\phi] (z) := \frac{\left(\frac{\delta}{\pi} \right)^{\frac{1}{2}}}{\Gamma(\delta+1)(1-z)^{\delta+1}} \int_0^{+\infty} \exp \left(-\frac{x}{2} \left(\frac{1+z}{1-z} \right) \right) \phi(x) x^\delta dx \quad (1.2)$$

where

$$\mathcal{A}^{(\delta+1)}(\mathbb{D}) := \left\{ \psi \text{ analytic on } \mathbb{D}, \int_{\mathbb{D}} |\psi(z)|^2 (1-|z|^2)^{\delta-1} d\mu(z) < +\infty \right\} \quad (1.3)$$

denotes the weighted Bergman space on the unit disk $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$ and $d\mu(z)$ being the Lebesgue measure on it. Thus, setting $\delta = \sigma - 1 > 0$ and using the isometry form $L^2(\mathbb{R}_+, dx)$ onto $L^2(\mathbb{R}_+, \Gamma^{-1}(\sigma) x^{\sigma-1} dx)$, one extends B_δ to the transform

$$\mathcal{B}_\sigma^\Pi : L^2(\mathbb{R}_+, dx) \rightarrow \mathcal{A}^\sigma(\mathbb{D}) \quad (1.4)$$

defined by

$$\mathcal{B}_\sigma^\Pi [\phi] (z) := \frac{\left(\frac{(\sigma-1)}{\pi\Gamma(\sigma)} \right)^{\frac{1}{2}}}{(1-z)^\sigma} \int_0^{+\infty} \exp \left(-\frac{x}{2} \left(\frac{1+z}{1-z} \right) \right) \phi(x) x^{\frac{1}{2}(\sigma-1)} dx. \quad (1.5)$$

The involved kernel function in (1.5) corresponds to the generating function of Laguerre polynomials [2]. The latter ones turn out to be fundamental pieces in expressing wave functions of the eigenstates of the pseudoharmonic oscillator Hamiltonian (see [3] and references therein). Now, after being observed that the role of the Laguerre polynomials can be replaced by the continuous dual Hahn polynomials [4] which are involved in the wave functions of the eigenstates of a relativistic pseudoharmonic oscillator Hamiltonian [5], here we propose a "relativistic" version of the second Bargmann transform (1.4), which also

will include a generalization of the arrival space \mathcal{A}_m^σ in (1.4). The latter one will be replaced by the eigenspace ([6, 7]):

$$\mathcal{A}_m^\sigma(\mathbb{D}) := \left\{ \psi : \mathbb{D} \rightarrow \mathbb{C}, \Delta_\sigma \psi = \epsilon_m^\sigma \psi, \int_{\mathbb{D}} |\psi(z)|^2 (1 - z\bar{z})^{\sigma-2} d\mu(z) < +\infty \right\} \quad (1.6)$$

of the second order differential operator

$$\Delta_\sigma := -4(1 - z\bar{z}) \left((1 - z\bar{z}) \frac{\partial^2}{\partial z \partial \bar{z}} - \sigma \bar{z} \frac{\partial}{\partial \bar{z}} \right) \quad (1.7)$$

with the eigenvalue (*hyperbolic Landau level*):

$$\epsilon_m^\sigma := 4m(\sigma - 1 - m), m = 0, 1, 2, \dots, \left\lfloor \frac{\sigma - 1}{2} \right\rfloor, \quad (1.8)$$

where $[x]$ denotes the greatest integer less than x . The operator in (1.6) can be unitarily intertwined to represent the Schrödinger operator of a charged particle evolving in the Poincaré disk under influence of a uniform magnetic field with a strength proportional to σ . For $m = 0$, the space $\mathcal{A}_0^\sigma(\mathbb{D})$ in (1.5) coincides with the Bergman space $\mathcal{A}^\sigma(\mathbb{D})$ in (1.3) where $\delta = \sigma - 1$. In this paper, we precisely construct a family of integral transforms of the form $\mathcal{B}_{c,m}^\Pi : L^2(\mathbb{R}_+) \rightarrow \mathcal{A}^{2(\gamma+m)}(\mathbb{D})$ defined by

$$\begin{aligned} \mathcal{B}_{c,m}^\Pi[f](z) &:= \frac{(2\gamma - 1)^{\frac{1}{2}}}{\sqrt{\pi m!} (1 - z)^{2\gamma}} \frac{\sqrt{2\Gamma(m + 2\gamma)} (i)^\gamma}{\Gamma(2\gamma) \Gamma(\gamma + \frac{1}{2})} \left(\frac{\bar{z} - 1}{(1 - z)(1 - z\bar{z})} \right)^m \\ &\times \int_0^{+\infty} \frac{\Gamma^2(\gamma + i\tilde{\zeta}) (c^{-4})^{i\tilde{\zeta}}}{\Gamma(i\tilde{\zeta})} F_5 \left(\begin{matrix} \gamma + i\tilde{\zeta}, & \gamma - i\tilde{\zeta} : & 2\gamma + m \\ & \gamma + \frac{1}{2} : & 2\gamma \end{matrix} \middle| \frac{1 - \bar{z}z}{(1 - \bar{z})(z - 1)}, \frac{1}{1 - \bar{z}} \right) f(\tilde{\zeta}) d\tilde{\zeta}, \end{aligned} \quad (1.9)$$

where $\sigma = 2(\gamma + m)$, $c > 0$, $\gamma = \gamma(c) = (1 + \sqrt{1 + 2c^4})/2$ and F_5 is a special Appel-Kampé de Fériet hypergeometric function [8]. Our method in constructing the transform (1.8) is based on a coherent states analysis by adopting a general probabilistic scheme “à la Gazeau” [9]. In the analytic case which corresponds to the particular value $m = 0$, we prove that the transform (1.9) reduces to the following one

$$\mathcal{B}_{c,0}^\Pi : L^2(\mathbb{R}_+) \rightarrow \mathcal{A}^{2\gamma}(\mathbb{D}) \quad (1.10)$$

defined by

$$\begin{aligned} \mathcal{B}_{c,0}^\Pi[f](z) &= \frac{\sqrt{2} (-i)^\gamma \left(\frac{2\gamma - 1}{\pi \Gamma(2\gamma)} \right)^{\frac{1}{2}}}{\Gamma(\gamma + \frac{1}{2}) (1 - z)^\gamma} \\ &\times \int_0^{+\infty} \frac{\Gamma^2(\gamma - i\tilde{\zeta}) \Gamma^{-1}(-i\tilde{\zeta})}{(c^{-4})^{i\tilde{\zeta}} (1 - z)^{i\tilde{\zeta}}} {}_2F_1 \left(\begin{matrix} \gamma - i\tilde{\zeta}, \frac{1}{2} - i\tilde{\zeta} \\ \gamma + \frac{1}{2} \end{matrix} \middle| z \right) f(\tilde{\zeta}) d\tilde{\zeta}, \end{aligned} \quad (1.11)$$

where ${}_2F_1(\cdot)$ denotes Gauss hypergeometric function [10]).

The paper is organized as follows. In Section 2, we recall briefly some needed tools from the spectral theory of the σ -weight Maass Laplacians on the Poincaré disk. Section 3 deals with the construction of a set of coherent states in the framework of a probabilistic Hilbertian schem without specifying the corresponding Hamiltonian system. In section 4 we summarize some required information on a relativistic model for the pseudoharmonic oscillator. In section 5 we particularize the constructed coherent states for the relativistic

pseudoharmonic oscillator and we obtain expressions for their wave functions in an explicit way. Section 6 is devoted to establish the corresponding coherent states transforms.

2. MAASS LAPLACIANS ON THE POINCARÉ DISK

Let $\mathbb{D} = \{z \in \mathbb{C}, |z| < 1\}$ be unit disk endowed with its usual Kähler metric $ds^2 = -\partial\bar{\partial}\text{Log}(1 - z\bar{z}) dz \otimes d\bar{z}$. The Bergman distance on \mathbb{D} is given by

$$\cosh^2 d(z, w) = \frac{(1 - z\bar{w})(1 - \bar{z}w)}{(1 - z\bar{z})(1 - w\bar{w})} \quad (2.1)$$

and the volume element reads

$$d\mu(z) = \frac{1}{(1 - z\bar{z})^2} dv(z) \quad (2.2)$$

with the Lebesgue measure $dv(z)$. Let us consider the differential 1-form on \mathbb{D} defined by $\theta = -i(\partial - \bar{\partial})\text{Log}(1 - z\bar{z})$ to which the Schrödinger operator

$$H_\sigma := \left(d + i\frac{\sigma}{2}\text{ext}(\theta)\right)^* \left(d + i\frac{\sigma}{2}\text{ext}(\theta)\right) \quad (2.3)$$

can be associated. Here $\sigma \geq 0$ is a fixed number, d denotes the usual exterior derivative on differential forms on \mathbb{D} and $\text{ext}(\theta)$ is the exterior multiplication by θ while the symbol $*$ stands for the adjoint operator with respect to the Hermitian scalar product induced by the Bergman metric ds^2 on differential forms. Actually, the operator H_σ is acting on the Hilbert space $L^2(\mathbb{D}, d\mu(z))$ and can be unitarily intertwined as

$$(1 - z\bar{z})^{\frac{1}{2}\sigma} \Delta_\sigma (1 - z\bar{z})^{-\frac{1}{2}\sigma} = H_\sigma \quad (2.4)$$

in terms of the second order differential operator Δ_σ introduced in (1.7). The latter one is acting on the Hilbert space $L^{2,\sigma}(\mathbb{D}) = L^2\left(\mathbb{D}, (1 - z\bar{z})^{\sigma-2} dv(z)\right)$. Note that this operator is an elliptic densely defined operator on $L^{2,\sigma}(\mathbb{D})$ and admits a unique self-adjoint realization that we denote also by Δ_σ . The part of its spectrum is not empty if and only if $\sigma > 1$. This discrete part consists of eigenvalues occurring with infinite multiplicities and having the expression $\epsilon_m^\sigma = 4m(\sigma - m - 1)$ in (1.8) for varying $m = 0, 1, \dots, [(\sigma - 1)/2]$. Moreover, it is well known ([6, 7, 11]) that the functions given in terms of Jacobi polynomials [2] by

$$\Phi_k^{\sigma,m}(z) := \sqrt{\frac{(\sigma - 2m - 1)\Gamma(\sigma - m)k!}{\pi m!\Gamma(\sigma - 2m + k)}} \frac{(-1)^k \bar{z}^{m-k}}{(1 - z\bar{z})^m} P_k^{(m-k, \sigma-2m-1)}(1 - 2z\bar{z}) \quad (2.5)$$

constitute an orthonormal basis of the eigenspace

$$\mathcal{A}_m^\sigma(\mathbb{D}) := \left\{ \phi \in L^{2,\sigma}(\mathbb{D}), \Delta_\sigma \phi = \epsilon_m^\sigma \phi \right\}. \quad (2.6)$$

of Δ_σ associated with the eigenvalue ϵ_m^σ in (2.6). Finally, the L^2 -eigenspace $\mathcal{A}_0^\sigma(\mathbb{D}) = \{\phi \in L^{2,\sigma}(\mathbb{D}), \Delta_\sigma \phi = 0\}$ corresponding to $m = 0$ and associated to $\epsilon_0^\sigma = 0$ in (2.7) reduces further to the weighted Bergman space consisting of holomorphic functions $\phi: \mathbb{D} \rightarrow \mathbb{C}$ with the growth condition

$$\int_{\mathbb{D}} |\phi(z)|^2 (1 - z\bar{z})^{\sigma-2} dv(z) < +\infty. \quad (2.7)$$

This is why the eigenspaces in (2.6) are also called generalized Bergman spaces on the complex unit disk.

Remark 2.1. *The spectral analysis of Δ_σ have been studied by many authors, see [6] and references therein and it can also be obtained from the σ -weight Maass Laplacian $y^2 (\partial_x^2 + \partial_y^2) - i\sigma y \partial_x$ on the Poincaré upper half-plane [12]. The condition $\sigma > 1$ ensuring the existence of the eigenvalues ϵ_m^σ in (1.8) should implies that the magnetic field $B = \sigma \Omega(z)$, where Ω stands for the Khäler 2-form on \mathbb{D} , has to be strong enough to capture the particle in a closed orbit. If this condition is not fulfilled the motion will be unbounded and the classical orbit of the particle will intercept the disk boundary whose points stands for $\{\infty\}$ which means escaping to infinity (see [13, p.189]).*

3. COHERENT STATES IN A HILBERTIAN PROBABILISTIC SCHEM

The negative binomial states [14] are labeled by points $z \in \mathbb{D}$ and are of the form

$$|z, \sigma, 0\rangle := (1 - z\bar{z})^{\frac{1}{2}\sigma} \sum_{k=0}^{+\infty} \sqrt{\frac{\Gamma(\sigma + k)}{\Gamma(\sigma) k!}} z^k |\psi_k\rangle \quad (3.1)$$

where $\sigma > 1$ is a fixed parameter and the kets $|\psi_k\rangle$ are for instance elements of an abstract Hilbert space \mathcal{H} . Their photon-counting probability distribution is given by

$$\Pr(X = k) := |\langle \psi_k | z, \sigma, 0 \rangle|^2 = (1 - z\bar{z})^\sigma (z\bar{z})^k \frac{\Gamma(\sigma + k)}{\Gamma(\sigma) k!}$$

which obeys the negative binomial probability distribution [15]. Observe that the coefficients in the superposition (3.1):

$$\Phi_k^{\sigma,0}(z) := \sqrt{\frac{\Gamma(\sigma + k)}{\pi \Gamma(\sigma) k!}} z^k, k = 0, 1, 2, \dots, \quad (3.2)$$

constitute an orthonormal basis of the eigenspace $\mathcal{A}_0^\sigma(\mathbb{D})$ associated with the first eigenvalue $\epsilon_0^\sigma = 0$ and consisting of analytic functions on \mathbb{D} with the growth condition (2.7). For instance, let $\sigma > 1$ and $m = 0, 1, \dots, [(\sigma - 1)/2]$ be fixed parameters and let $\{|\psi_k\rangle\}_{k=0}^\infty$ be a set of Fock states in a Hilbert space \mathcal{H} . Then, adopting the Hilbertian probabilistic schem of coherent sates in ([9, p.74, Eq. (58)]), we state the following.

Definition 3.1. *A class of coherent states can be defined as*

$$|z, \sigma, m\rangle = (\mathcal{N}_{\sigma,m}(z))^{-\frac{1}{2}} \sum_{k=0}^{+\infty} \overline{\Phi_k^{\sigma,m}(z)} |\psi_k\rangle \quad (3.3)$$

where $\mathcal{N}_{\sigma,m}(z)$ is a normalization factor and $\{\Phi_k^{\sigma,m}(z)\}_{k=0}^\infty$ is the orthonormal basis (2.5) of the generalized Bergman space $\mathcal{A}_m^\sigma(\mathbb{D})$.

Now, one of the important task to do is to determine is the overlap relation between two coherent states.

Proposition 3.2. *Let $\sigma > 1$ and $m = 0, 1, \dots, [(\sigma - 1)/2]$. Then, for every $z, w \in \mathbb{D}$, the overlap relation between two coherent states is given through the scalar product*

$$\begin{aligned} \langle w, \sigma, m | z, \sigma, m \rangle_{\mathcal{H}} &= \frac{(\sigma - 2m - 1) \Gamma(\sigma - m) (\mathcal{N}(z) \mathcal{N}(w))^{-\frac{1}{2}}}{\pi m! (-1)^m \Gamma(\sigma - 2m) (1 - z\bar{w})^\sigma} \\ &\times \left(\frac{(1 - z\bar{w})(1 - \bar{w}z)}{(1 - z\bar{z})(1 - w\bar{w})} \right)^m \cdot {}_2F_1 \left(-m, \sigma - m, \sigma - 2m; \frac{(1 - z\bar{z})(1 - w\bar{w})}{(1 - z\bar{w})(1 - \bar{w}z)} \right) \end{aligned} \quad (3.4)$$

where ${}_2F_1$ is a terminating Gauss hypergeometric sum.

Proof. In view of Eq. (3.3), the scalar product of two coherent states $|z, \sigma, m\rangle$ and $|w, \sigma, m\rangle$ in \mathcal{H} reads

$$\begin{aligned} \langle w, \sigma, m | z, \sigma, m \rangle_{\mathcal{H}} &= (\mathcal{N}_{\sigma, m}(z) \mathcal{N}_{\sigma, m}(w))^{-\frac{1}{2}} \sum_{k=0}^{+\infty} \Phi_k^{\sigma, m}(z) \overline{\Phi_k^{\sigma, m}(w)} \\ &= \frac{(\sigma - 2m - 1) (\mathcal{N}(z) \mathcal{N}(w))^{-\frac{1}{2}}}{\pi ((1 - z\bar{z})(1 - w\bar{w}))^m} \mathcal{S}_{z, w}^{\sigma, m}, \end{aligned} \quad (3.5)$$

where

$$\begin{aligned} \mathcal{S}_{z, w}^{\sigma, m} &= \frac{\Gamma(\sigma - m) (\bar{z}w)^m}{m! \Gamma(\sigma - 2m)} \sum_{k=0}^{+\infty} \frac{k!}{(\sigma - 2m)_k} \left(\frac{1}{\bar{z}w} \right)^k \\ &\quad \times P_k^{(m-k, \sigma-2m-1)}(1 - 2z\bar{z}) P_k^{(m-k, \sigma-2m-1)}(1 - 2w\bar{w}). \end{aligned} \quad (3.6)$$

Making use of the following identity due to A. Sirvastava and A. B. Rao ([16, p.1329]):

$$\begin{aligned} \sum_{n=0}^{+\infty} \frac{n! t^n}{(1 + \alpha)_n} P_n^{(\gamma-n, \alpha)}(x) P_n^{(\gamma-n, \alpha)}(y) &= \left(1 - \frac{1}{4} (x-1)(y-1)t \right)^{1+\gamma+\alpha} \\ &\quad \times (1-t)^\gamma \cdot {}_2F_1 \left(1 + \gamma + \alpha, -\gamma, 1 + \alpha; \frac{-(x+1)(y+1)t}{(1-t)(4-(x-1)(y-1)t)} \right) \end{aligned} \quad (3.7)$$

for $n = k, t = 1/\bar{z}w, \gamma = m, \alpha = \sigma - 2m - 1, x = 1 - 2z\bar{z}$ and $y = 1 - 2w\bar{w}$, we obtain, after calculations, the expression

$$\begin{aligned} \mathcal{S}_{z, w}^{\sigma, m} &= \frac{\Gamma(\sigma - m) (-1)^m ((1 - z\bar{w})(1 - \bar{w}z))^m}{m! \Gamma(\sigma - 2m) (1 - z\bar{w})^\sigma} \\ &\quad \times {}_2F_1 \left(-m, \sigma - m, \sigma - 2m; \frac{(1 - z\bar{z})(1 - w\bar{w})}{(1 - z\bar{w})(1 - \bar{w}z)} \right). \end{aligned} \quad (3.8)$$

Returning back to Eq. (3.5) and inserting the expression (3.8) we arrive at the announced formula. \square

Corollary 3.3. . The normalization factor in (3.3) is given by

$$\mathcal{N}_{\sigma, m}(z) = \frac{(\sigma - 2m - 1)}{\pi (1 - z\bar{z})^\sigma}, \quad (3.9)$$

for every $z \in \mathbb{D}$.

Proof. We first make appeal to the relation ([17, p.212]):

$${}_2F_1 \left(-n, n + \kappa + \varrho + 1, 1 + \kappa; \frac{1 - \tau}{2} \right) = \frac{n! \Gamma(1 + \kappa)}{\Gamma(1 + \kappa + n)} P_n^{(\kappa, \varrho)}(\tau) \quad (3.10)$$

connecting the ${}_2F_1$ -sum with the Jacobi polynomial for the parameters $n = m, \kappa = \sigma - 2m - 1, \varrho = 0$ and the variable

$$\tau = 1 - 2 \frac{(1 - z\bar{z})(1 - w\bar{w})}{(1 - z\bar{w})(1 - \bar{w}z)} \quad (3.11)$$

to rewrite Eq. (3.4) as

$$\begin{aligned} \sqrt{\mathcal{N}_{\sigma,m}(z)\mathcal{N}_{\sigma,m}(\bar{w})} &= \frac{(-1)^m (\sigma - 2m - 1) (1 - z\bar{w})^{-\sigma}}{\pi \langle z, \sigma, m | z, \sigma, m \rangle_{\mathcal{H}}} \\ &\times P_m^{(\sigma-2m-1,0)} \left(1 - 2 \frac{(1 - z\bar{z})(1 - w\bar{w})}{(1 - z\bar{w})(1 - \bar{w}z)} \right). \end{aligned} \quad (3.12)$$

The factor $\mathcal{N}_{\sigma,m}(z)$ should be such that $\langle z, \sigma, m | z, \sigma, m \rangle_{\mathcal{H}} = 1$. So that we put $z = w$ in (3.12) and we use the well known symmetry identity satisfied by the Jacobi polynomials $P_m^{(\gamma,\varrho)}(\xi) = (-1)^m P_m^{(\varrho,\gamma)}(-\xi)$ to obtain the expression

$$\mathcal{N}_{\sigma,m}(z) = \frac{(\sigma - 2m - 1)}{\pi (1 - z\bar{z})^\sigma} P_m^{(0,\sigma-2m-1)}(1) \quad (3.13)$$

Finally, we apply the fact that ([17, p.209]):

$$P_n^{(\alpha,\varrho)}(1) = \frac{\Gamma(n + \alpha + 1)}{n! \Gamma(\alpha + 1)} \quad (3.14)$$

in the case of $\alpha = 0, n = m$ and $\varrho = \sigma - 2m - 1$. This ends the proof. \square

Proposition 3.4. *Let $\sigma > 1$ and $m = 0, 1, \dots, [(\sigma - 1)/2]$. Then, the states in (3.3) satisfy the following resolution of the identity*

$$\mathbf{1}_{\mathcal{H}} = \int_{\mathbb{D}} |z, \sigma, m\rangle \langle z, \sigma, m| d\mu_{\sigma,m}(z) \quad (3.15)$$

where $\mathbf{1}_{\mathcal{H}}$ is the identity operator, $d\mu_{\sigma,m}(z)$ is a measure given by

$$d\mu_{\sigma,m}(z) := \pi^{-1} (\sigma - 2m - 1) (1 - z\bar{z})^{-2} dv(z), \quad (3.16)$$

and $dv(z)$ being the Lebesgue measure on \mathbb{D} .

Proof. Let us assume that the measure takes the form $d\mu_{\sigma,m}(z) = \mathcal{N}_{\sigma,m}(z) \Omega(z) dv(z)$ where $\Omega(z)$ is an auxiliary density to be determined. Let $\varphi \in \mathcal{H}$ and let us start by writing the following action

$$\mathcal{O}[\varphi] := \left(\int_{\mathbb{D}} |z, \sigma, m\rangle \langle z, \sigma, m| d\mu_{\sigma,m}(z) \right) [\varphi] \quad (3.17)$$

$$= \int_{\mathbb{D}} \langle \varphi | z, \sigma, m\rangle \langle z, \sigma, m| d\mu_{\sigma,m}(z). \quad (3.18)$$

Making use Eq. (3.3), we obtain successively

$$\mathcal{O}[\varphi] = \int_{\mathbb{D}} \langle \varphi | (\mathcal{N}_{\sigma,m}(z))^{-\frac{1}{2}} \sum_{k=0}^{+\infty} \Phi_k^{\sigma,m}(z) | \psi_k \rangle \langle z, \sigma, m| d\mu_{\sigma,m}(z) \quad (3.19)$$

$$= \left(\sum_{j,k=0}^{+\infty} \int_{\mathbb{D}} \overline{\Phi_j^{\sigma,m}(z)} \Phi_k^{\sigma,m}(z) | \psi_k \rangle \langle \psi_j | (\mathcal{N}_{\sigma,m}(z))^{-1} d\mu_{\sigma,m}(z) \right) [\varphi]. \quad (3.20)$$

We replace the measure $d\mu_{\sigma,m}(z)$ by the expression $\mathcal{N}_{\sigma,m}(z) \Omega(z) dv(z)$, then Eq. (3.20) can be written without φ as follows

$$\mathcal{O} = \sum_{j,k=0}^{+\infty} \left[\int_{\mathbb{D}} \overline{\Phi_j^{\sigma,m}(z)} \Phi_k^{\sigma,m}(z) \Omega(z) dv(z) \right] | \psi_j \rangle \langle \psi_k |. \quad (3.21)$$

Therefore, we need to have

$$\int_{\mathbb{D}} \overline{\Phi_j^{\sigma,m}(z)} \Phi_k^{\sigma,m}(z) \Omega(z) d\nu(z) = \delta_{jk}. \quad (3.22)$$

For this we recall the orthogonality relation of the $\Phi_k^{\sigma,m}(z)$ in the Hilbert space $L^{2,\sigma}(\mathbb{D})$, which reads

$$\int_{\mathbb{D}} \overline{\Phi_j^{\sigma,m}(z)} \Phi_k^{\sigma,m}(z) (1 - z\bar{z})^{\sigma-2} d\nu(z) = \delta_{jk}. \quad (3.23)$$

This suggests us to set $\Omega(z) := (1 - z\bar{z})^{\sigma-2}$. Therefore, we get that

$$d\mu_{\sigma,m}(z) = \frac{(\sigma - 2m - 1)}{\pi (1 - z\bar{z})^2} d\nu(z). \quad (3.24)$$

we arrive at the announced expression of the measure. Therefore, Eq. (3.21) reduces to

$$\mathcal{O} = \sum_{j,k=0}^{+\infty} \delta_{jk} |\psi_j\rangle \langle \psi_k| = \mathbf{1}_{\mathcal{H}}. \quad (3.25)$$

The proof is finished. \square

Proposition 3.5. *Let $\sigma > 1$ and $m = 0, 1, \dots, [(\sigma - 1)/2]$. Then, the states $|z, \sigma, m\rangle$ satisfy the continuity property with respect to the label $z \in \mathbb{D}$. That is, the norm of the difference of two states*

$$d_{\sigma,m}(z, w) := \|(|z, \sigma, m\rangle - |w, \sigma, m\rangle)\|_{\mathcal{H}} \quad (3.26)$$

goes to zero whenever $z \rightarrow w$.

Proof. By using the fact that any state $|z, \sigma, m\rangle$ is normalized by the factor given in (3.9), direct calculations enable us to write the square of the quantity in (3.26) as

$$d_{\sigma,m}^2(z, w) = 2(1 - \Re(\langle z, \sigma, m | w, \sigma, m \rangle)). \quad (3.27)$$

Next, we use of the expression of the scalar product in (3.9) form which it is clear that the overlap takes the value 1 as $z \rightarrow w$ and consequently $d_{\sigma,m}(z, w) \rightarrow 0$. \square

We have verified that the basic minimum properties for the constructed states to be considered as coherent states are satisfied. Namely, the conditions which have been formulated by Klauder [18]: (a) the continuity of labeling, (b) the fact that these states are normalizable but not orthogonal and (c) these states fulfilled the resolution of the identity with a positive weight function. As we can see, these coherent states are independent of the basis $|\psi_k\rangle$ we use and the only condition which is implicitly fulfilled is the orthonormality of the basis vectors of \mathcal{H} . But if we want to attach these coherent states to a concrete quantum system then a Hamiltonian operator should be specified together with a corresponding explicit eigenstates basis. This will be the goal of the next section.

4. A RELATIVISTIC PSEUDOHARMONIC OSCILLATOR

In this section, we recall some needed results which have been developed in [5], where the authors considered a model for the relativistic pseudo-harmonic oscillator with the following interaction potential

$$U_{m_*,\omega,\lambda,g}(x) := \left(\frac{1}{2} m_* \omega^2 x (x + i\lambda) + \frac{g}{x(x + i\lambda)} \right) e^{i\beta \partial_x} \quad (4.1)$$

where ω is a frequency, $g \geq 0$ is a real quantity and $\lambda = \hbar/m_*c$ denotes the Compton wavelength defined by the ratio of Planck's constant \hbar by the mass m_* times the speed

of light c . The corresponding stationary Schrödinger equation is described by the finite-difference equation

$$\left(mc^2 \cosh i\beta \partial_{\xi} + U_{m_*, \omega, \beta, g}(x) \right) \varphi(\xi) = E \varphi(\xi) \quad (4.2)$$

with the boundary conditions for the wave function $\varphi(0) = 0$ and $\varphi(\infty) = 0$. As in [5], we will restrict ourself to the interval $0 \leq x < \infty$ and in terms of dimensionless variable $\xi = x/\lambda$ and parameters $\omega_0 = \hbar\omega/m_*c^2, g_0 = m_*g/\hbar^2$ the equation (4.2) takes the form

$$\left(\cosh i\partial_{\xi} + \frac{1}{2}\omega_0^2 \xi (\xi + i) e^{i\partial_{\xi}} + \frac{g_0}{\xi (\xi + i)} \right) \varphi(\xi) = \frac{E}{m_*c^2} \varphi(\xi) \quad (4.3)$$

The authors in [5] have obtained the energy spectrum of the Schrödinger operator in (4.2) as

$$E_k := \hbar\omega (2k + \alpha_+ + \alpha_-); \quad k = 0, 1, 2, \dots, \quad (4.4)$$

where

$$2\alpha_{\pm} - 1 = \sqrt{1 + \frac{2}{\omega_0^2} \left(1 \pm \sqrt{1 - 8g_0\omega_0^2} \right)}, \quad (4.5)$$

For our purpose to ensure that E_k are real we choose $1 - 8g_0\omega_0^2 = 0$ which means, in system of units $\hbar = m_* = \omega = 1$, the choice $g = c^4/8$. In this case $\alpha_+ = \alpha_- = \gamma_c = (1 + \sqrt{1 + 2c^4})/2 \equiv \gamma$. So that we will be concerned with eigenstates of the form ([5]):

$$\varphi_k^{\gamma}(\xi) := \frac{\sqrt{2}i^{\gamma} (c^{-4})^{i\xi} \Gamma^2(\gamma + i\xi) (\Gamma(i\xi))^{-1}}{\Gamma(k + \gamma + \frac{1}{2}) \sqrt{k! \Gamma(k + 2\gamma)}} S_k \left(\xi^2; \gamma, \gamma, \frac{1}{2} \right) \quad (4.6)$$

where $S_k(\xi^2; a, b, c)$ denotes the continuous dual Hahn polynomial ([4, p.331]), which can be defined in terms of the ${}_3F_2$ -sum as

$$S_n \left(\xi^2; a, b, c \right) := (a + b)_n (a + c)_n \cdot {}_3F_2 \left(\begin{matrix} -n, a + i\xi, a - i\xi \\ a + b, a + c \end{matrix}; 1 \right). \quad (4.7)$$

Finally, we note that the wave functions in (4.6) satisfy the relations:

$$\int_0^{+\infty} \varphi_k^{\gamma}(\xi) \overline{\varphi_j^{\gamma}(\xi)} d\xi = \delta_{k,j} \quad (4.8)$$

which means that they constitute an orthonormalized system in the Hilbert space $L^2(\mathbb{R}_+, d\xi)$.

5. A FAMILY OF COHERENT STATES FOR THE RELATIVISTIC PSEUDOHARMONIC OSCILLATOR

We now adopt the Hilbertian probabilistic schem discussed above in Section 3 to define a class of coherent states as follows.

Definition 5.1. For $\sigma > 1$ and $m = 0, 1, \dots, \left[\frac{\sigma-1}{2} \right]$ a class of coherent states for the relativistic pseudoharmonic oscillator (4.2) are defined by

$$\varphi_{z, \sigma, \gamma, m}(\cdot) := (\mathcal{N}_{\sigma, m}(z))^{-\frac{1}{2}} \sum_{k=0}^{+\infty} \overline{\Phi_k^{\sigma, m}(z)} | \varphi_k^{\gamma}(\cdot) > \quad (5.1)$$

where $\mathcal{N}_{\sigma, m}(z)$ is the factor in (3.9), $\Phi_k^{\sigma, m}(z)$ are given by (2.5) and $\varphi_k^{\gamma}(\cdot)$ are the eigenstates in (4.6).

Proposition 5.2. *Let $\sigma = 2(m + \gamma)$ and $z \in \mathbb{D}$ be a fixed labeling point. Then, the wave functions of the states (5.1) are of the form*

$$\begin{aligned} \varphi_{z,\gamma,m}(\xi) &= \frac{\sqrt{2\Gamma(m+2\gamma)} i^\gamma (c^{-4})^{i\xi} \Gamma^2(\gamma + i\xi)}{\Gamma(2\gamma) \Gamma\left(\gamma + \frac{1}{2}\right) \sqrt{m!} \Gamma(i\xi)} \\ &\times \frac{(1 - z\bar{z})^\gamma}{(1 - z)^{2\gamma}} \left(\frac{\bar{z} - 1}{1 - z}\right)^m F_5 \left(\begin{matrix} \gamma + i\xi, & \gamma - i\xi : & 2\gamma + m \\ & \gamma + \frac{1}{2} : & 2\gamma \end{matrix} \middle| \frac{1 - \bar{z}z}{(1 - \bar{z})(z - 1)}, \frac{1}{1 - \bar{z}} \right) \end{aligned} \quad (5.2)$$

for any $\xi \in \mathbb{R}_+$, where F_5 is a special Appel-Kampé de Fériet hypergeometric function.

Proof. We start from Eq. (5.1) by replacing the coefficients $\overline{\Phi_k^{\sigma,m}(z)}$ by their expressions taking into account Eq. (2.5). This leads to the expression

$$\begin{aligned} \varphi_{z,\gamma,m}(\xi) &= \left(\frac{(\sigma - 2m - 1)}{\pi(1 - z\bar{z})^\sigma} \right)^{-\frac{1}{2}} \sum_{k=0}^{\infty} \sqrt{\frac{(\sigma - 2m - 1) k! \Gamma(\sigma - m)}{\pi m! \Gamma(\sigma - 2m + k)}} \\ &\times (1 - z\bar{z})^{-m} (-1)^k z^{m-k} P_k^{(m-k, \sigma-2m-1)} (1 - 2z\bar{z}) \varphi_k^\gamma(\xi). \end{aligned} \quad (5.3)$$

Next, introducing the variable $u := 2z\bar{z} - 1$ and inserting the expression of $\varphi_k^\gamma(\xi)$ in Eq. (5.3), we obtain that

$$\begin{aligned} \varphi_{z,\gamma,m}(\xi) &= z^m (1 - z\bar{z})^{\frac{1}{2}\sigma - m} \sqrt{\frac{2\Gamma(\sigma - m)}{m!}} i^\gamma (c^{-4})^{i\xi} \frac{\Gamma^2(\gamma + i\xi)}{\Gamma(i\xi)} \\ &\times \sum_{k=0}^{\infty} \sqrt{\frac{k!}{\Gamma(\sigma - 2m + k)}} \frac{P_k^{(\sigma-2m-1, m-k)}(u) S_k(\xi^2, \gamma, \gamma, 1/2)}{z^k \Gamma\left(k + \gamma + \frac{1}{2}\right) \sqrt{k! \Gamma(k + 2\gamma)}}. \end{aligned} \quad (5.4)$$

We use the notation $\sigma - 2m = 2\gamma$ and we focus on the sum in (5.4):

$$\mathfrak{S} := \sum_{k=0}^{\infty} \sqrt{\frac{k!}{\Gamma(2\gamma + k)}} \frac{P_k^{(2\gamma-1, m-k)}(u) S_k(\xi^2, \gamma, \gamma, 1/2)}{z^k \Gamma\left(k + \gamma + \frac{1}{2}\right) \sqrt{k! \Gamma(k + 2\gamma)}}. \quad (5.5)$$

Next, we set $t := 1/z$ and we rewrite (5.5) in a simple form as

$$\mathfrak{S} = \sum_{k=0}^{\infty} t^k \frac{P_k^{(2\gamma-1, m-k)}(u) S_k(\xi^2, \gamma, \gamma, 1/2)}{\Gamma\left(k + \gamma + \frac{1}{2}\right) \Gamma(2\gamma + k)}. \quad (5.6)$$

Now, we need the definition of the continuous dual Hahn polynomials by the hypergeometric terminating ${}_3F_2$ -sum ([4, p.331]):

$$S_k(\xi^2, \gamma, \gamma, 1/2) = \frac{\Gamma(2\gamma + k) \Gamma\left(\gamma + \frac{1}{2} + k\right)}{\Gamma(2\gamma) \Gamma\left(\gamma + \frac{1}{2}\right)} {}_3F_2 \left(\begin{matrix} -k, \gamma + i\xi, \gamma - i\xi \\ 2\gamma, \gamma + \frac{1}{2} \end{matrix} \middle| 1 \right). \quad (5.7)$$

So that the sum (5.6) becomes

$$\mathfrak{S} = \frac{1}{\Gamma(2\gamma) \Gamma\left(\gamma + \frac{1}{2}\right)} \sum_{k=0}^{\infty} t^k P_k^{(2\gamma-1, m-k)}(u) {}_3F_2 \left(\begin{matrix} -k, \gamma + i\xi, \gamma - i\xi \\ 2\gamma, \gamma + \frac{1}{2} \end{matrix} \middle| 1 \right). \quad (5.8)$$

Now, we make use of the integral representation ([19, p.84]):

$${}_3F_2 \left(\begin{matrix} \alpha, \beta, \rho \\ \tau, \rho + \omega \end{matrix} \middle| 1 \right) = \frac{\Gamma(\rho + \omega)}{\Gamma(\rho) \Gamma(\omega)} \int_0^1 x^{\rho-1} (1-x)^{\omega-1} {}_2F_1 \left(\begin{matrix} \alpha, \beta \\ \tau \end{matrix} \middle| x \right) dx \quad (5.9)$$

provided that $\Re(\rho) > 0$, $\Re(\omega) > 0$ and $\Re(\tau + \omega - \alpha - \beta) > 0$ for the parameters $\alpha = -k, \beta = \gamma + i\zeta, \rho = \gamma - i\zeta, \tau = 2\gamma, \omega = \frac{1}{2} + i\zeta$ to rewrite the ${}_3F_2$ -sum in (5.8) as

$${}_3F_2 \left(\begin{matrix} -k, \gamma + i\zeta, \gamma - i\zeta \\ 2\gamma, \gamma + \frac{1}{2} \end{matrix} \middle| 1 \right) = \frac{\Gamma\left(\gamma + \frac{1}{2}\right)}{\Gamma(\gamma - i\zeta) \Gamma\left(\frac{1}{2} + i\zeta\right)} \times \int_0^1 x^{\gamma - i\zeta - 1} (1 - x)^{-\frac{1}{2} + i\zeta} {}_2F_1 \left(\begin{matrix} -k, \gamma + i\zeta \\ 2\gamma \end{matrix} \middle| x \right) dx. \quad (5.10)$$

Therefore, Eq. (5.8) transforms successively as

$$\mathfrak{S} = \frac{1}{\Gamma(2\gamma) \Gamma\left(\gamma + \frac{1}{2}\right)} \frac{\Gamma\left(\gamma + \frac{1}{2}\right)}{\Gamma(\gamma - i\zeta) \Gamma\left(\frac{1}{2} + i\zeta\right)} \sum_{k=0}^{\infty} t^k P_k^{(2\gamma-1, m-k)}(u) \quad (5.11)$$

$$\times \int_0^1 x^{\gamma - i\zeta - 1} (1 - x)^{-\frac{1}{2} + i\zeta} {}_2F_1 \left(\begin{matrix} -k, \gamma + i\zeta \\ 2\gamma \end{matrix} \middle| x \right) dx \\ = \frac{1}{\Gamma(2\gamma)} \int_0^1 \Phi(x, \zeta) \left(\sum_{k=0}^{\infty} t^k P_k^{(2\gamma-1, m-k)}(u) {}_2F_1 \left(\begin{matrix} -k, \gamma + i\zeta \\ 2\gamma \end{matrix} \middle| x \right) \right) dx, \quad (5.12)$$

where

$$\Phi(x, \zeta) := \frac{x^{\gamma - i\zeta - 1} (1 - x)^{-\frac{1}{2} + i\zeta}}{\Gamma(\gamma - i\zeta) \Gamma\left(\frac{1}{2} + i\zeta\right)}. \quad (5.13)$$

Now, we look closely at the sum in (5.12):

$$\Xi(x) := \sum_{k=0}^{\infty} t^k P_k^{(2\gamma-1, m-k)}(u) {}_2F_1 \left(\begin{matrix} -k, \gamma + i\zeta \\ 2\gamma \end{matrix} \middle| x \right). \quad (5.14)$$

We exploit the connection formula ([20, p.63]):

$$P_n^{(\alpha, \beta)}(u) = \left(\frac{1-u}{2} \right)^n P_n^{(-2n-\alpha-\beta-1, \beta)} \left(\frac{u+3}{u-1} \right) \quad (5.15)$$

for the parameters $\alpha = 2\gamma - 1, \beta = m - k$ and $n = k$ to rewrite the Jacobi polynomial in (5.14) as

$$P_k^{(2\gamma-1, m-k)}(u) = (1 - z\bar{z})^k P_k^{(-2\gamma-m-k, m-k)} \left(\frac{z\bar{z} + 1}{z\bar{z} - 1} \right). \quad (5.16)$$

Therefore Eq. (5.14) can be rewritten as

$$\Xi = \sum_{k=0}^{\infty} \left(\frac{1 - z\bar{z}}{z} \right)^k P_k^{(-2\gamma-m-k, m-k)} \left(\frac{z\bar{z} + 1}{z\bar{z} - 1} \right) {}_2F_1 \left(\begin{matrix} -k, \gamma + i\zeta \\ 2\gamma \end{matrix} \middle| x \right). \quad (5.17)$$

We introduce the variables

$$\theta := \frac{1 - z\bar{z}}{z}, V := \frac{z\bar{z} + 1}{z\bar{z} - 1} \quad (5.18)$$

in terms of which Eq. (5.17) also reads

$$\Xi = \sum_{k=0}^{\infty} \frac{(2\gamma)_k}{(2\gamma)_k} \theta^k P_k^{(-2\gamma-m-k, m-k)}(V) {}_2F_1 \left(\begin{matrix} -k, \gamma + i\zeta \\ 2\gamma \end{matrix} \middle| x \right). \quad (5.19)$$

We are now in position to apply the bilinear generating formula due to S. Saran ([21, p.14]):

$$\sum_{k=0}^{\infty} \frac{(b)_k}{(d)_k} \theta^k P_k^{(\alpha-k, \beta-k)}(V) \cdot {}_2F_1 \left(\begin{matrix} -k, c \\ b \end{matrix} \middle| y \right) \quad (5.20)$$

$$= (1-y)^{-c} F_8 \left(b, b, b, c, -\alpha, -\beta; b, d, d; \frac{y}{y-1}, -\frac{1}{2}(V+1)\theta, -\frac{1}{2}(V-1)\theta \right).$$

In our context $b = d = 2\gamma$, this implies that the Lauricella triple hypergeometric series F_8 , which is denoted F_G by S. Saran, reduces to the expression

$$\left[1 + \frac{(V+1)\theta}{2} \right]^\alpha \left[1 + \frac{(V-1)\theta}{2} \right]^\beta (1-y)^c \quad (5.21)$$

$$\times F_1 \left(c, -\alpha, -\beta; b; \frac{y(V+1)\theta}{2+(V+1)\theta'}, \frac{y(V-1)\theta}{2+(V-1)\theta'} \right)$$

in terms of the first F_1 Appell's hypergeometric function ([19], p.265). Therefore, in terms of our parameters, the sum (5.19) also has the following expression

$$\Xi = \left[1 + \frac{(V+1)\theta}{2} \right]^{-2\gamma-m} \left[1 + \frac{(V-1)\theta}{2} \right]^m \quad (5.22)$$

$$\times F_1 \left(\gamma + i\zeta, 2\gamma + m, -m; 2\gamma; \frac{x(V+1)\theta}{2+(V+1)\theta'}, \frac{x(V-1)\theta}{2+(V-1)\theta'} \right).$$

Now if we denote the parameters and arguments occurring the last Appell F_1 -sum respectively by $a = \gamma + i\zeta$, $b = 2\gamma + m$, $c = -m$, $d = 2\gamma$,

$$X = \frac{x(V+1)\theta}{2+(V+1)\theta} = \mu_z x, Y = \frac{x(V-1)\theta}{2+(V-1)\theta} = \nu_z x \quad (5.23)$$

with

$$\mu_z := \frac{(V+1)\theta}{2+(V+1)\theta} = \frac{\bar{z}}{\bar{z}-1}, \nu_z := \frac{(V-1)\theta}{2+(V-1)\theta} = \frac{1}{1-z}, \quad (5.24)$$

then this F_1 -sum can be presented as $F_1(a, b, c, d; X, Y)$ with a particularity here consisting on the fact that the parameters b, c and d satisfy $d = b + c$ so that it can be reduced to a Gauss hypergeometric function according to the transformation ([22]):

$$F_1(a, b, c, b+c; X, Y) = (1-Y)^{-a} \cdot {}_2F_1 \left(\begin{matrix} a, b \\ b+c \end{matrix} \middle| \frac{X-Y}{1-Y} \right). \quad (5.25)$$

Therefore, using (5.25), we obtain from (5.22) the following fact

$$F_1(\gamma + i\zeta, 2\gamma + m, -m; 2\gamma; \mu_z x, \nu_z x)$$

$$= (1 - \nu_z x)^{-a} \cdot {}_2F_1 \left(\begin{matrix} \gamma + i\zeta, 2\gamma + m \\ 2\gamma \end{matrix} \middle| \frac{(\mu_z - \nu_z)x}{1 - \nu_z x} \right). \quad (5.26)$$

We set

$$\tau_z := \mu_z - \nu_z = \frac{\bar{z}}{\bar{z}-1} - \frac{1}{1-z} = \frac{1 - \bar{z}z}{(1-z)(\bar{z}-1)}. \quad (5.27)$$

So that the sum in (5.22) becomes

$$\Xi(x) = \left[1 + \frac{(V+1)\theta}{2} \right]^{-2\gamma-m} \left[1 + \frac{(V-1)\theta}{2} \right]^m \quad (5.28)$$

$$\times (1 - \nu_z x)^{-\gamma-i\zeta} \cdot {}_2F_1 \left(\begin{matrix} \gamma + i\zeta, 2\gamma + m \\ 2\gamma \end{matrix} \middle| \frac{(\mu_z - \nu_z)x}{1 - \nu_z x} \right).$$

Next, to compute the prefactor in the last Eq. (5.28), in terms of the fixed labeling point $z \in \mathbb{D}$, we are of need of the equalities

$$1 - \bar{z} = 1 + \frac{(V+1)\theta}{2} \text{ and } \frac{z-1}{z} = 1 + \frac{(V-1)\theta}{2}. \quad (5.29)$$

in order to rewrite (5.28) as

$$\begin{aligned} \Xi(x) &= (1 - \bar{z})^{-2\gamma} z^{-m} \left(\frac{z-1}{1-\bar{z}} \right)^m \\ &\times (1 - \nu_z x)^{-\gamma - i\tilde{\zeta}} {}_2F_1 \left(\begin{matrix} \gamma + i\tilde{\zeta}, 2\gamma + m \\ 2\gamma \end{matrix} \middle| \frac{\tau_z x}{1 - \nu_z x} \right). \end{aligned} \quad (5.30)$$

Returning back to the sum in (5.12) and inserting $\Xi(t)$, we get that

$$\mathfrak{S} = \frac{1}{\Gamma(2\gamma) \Gamma(\gamma - i\tilde{\zeta}) \Gamma\left(\frac{1}{2} + i\tilde{\zeta}\right)} \int_0^1 x^{\gamma - i\tilde{\zeta} - 1} (1-x)^{-\frac{1}{2} + i\tilde{\zeta}} (\Xi(x)) dx. \quad (5.31)$$

Explicitly, this last quantity reads

$$\begin{aligned} \mathfrak{S} &= \frac{1}{\Gamma(2\gamma) \Gamma(\gamma - i\tilde{\zeta}) \Gamma\left(\frac{1}{2} + i\tilde{\zeta}\right)} (1 - \bar{z})^{-2\gamma} z^{-m} \left(\frac{z-1}{1-\bar{z}} \right)^m \\ &\times \int_0^1 x^{\gamma - i\tilde{\zeta} - 1} (1-x)^{-\frac{1}{2} + i\tilde{\zeta}} (1 - \nu_z x)^{-\gamma - i\tilde{\zeta}} {}_2F_1 \left(\begin{matrix} 2\gamma + m, \gamma + i\tilde{\zeta} \\ 2\gamma \end{matrix} \middle| \frac{\tau_z x}{1 - \nu_z x} \right) dx. \end{aligned} \quad (5.32)$$

A this stade we can make use of the integral representation due to S.K. Kulshreshtha ([8, p.137, Eq. (2.2)]), of a very special case of the Appel-Kampé de Fériet's hypergeometric function of two variables of higher order [23] as follows

$$F \left[\begin{matrix} 2 & c & d \\ 1 & a & b \\ 1 & e & . \\ 1 & a' & b \end{matrix} \middle| \begin{matrix} \chi \\ \zeta \end{matrix} \right] \equiv F_5 \left(\begin{matrix} c, & d : & a \\ . & e : & a' \end{matrix} \middle| \chi, \zeta \right) \quad (5.33)$$

$$= \frac{\Gamma(e)}{\Gamma(d) \Gamma(e-d)} \int_0^1 t^{d-1} (1-t)^{e-d-1} (1-\zeta t)^{-c} {}_2F_1 \left(\begin{matrix} a, c \\ a' \end{matrix} \middle| \frac{\chi t}{1-\zeta t} \right) dt \quad (5.34)$$

for the parameters $d = \gamma - i\tilde{\zeta}, e = \frac{1}{2} + \gamma, c = \gamma + i\tilde{\zeta}, a = 2\gamma + m, a' = 2\gamma, \zeta = \nu_z, \chi = \tau_z$ and $t = x$. So that the integral occurring in (5.32) reads

$$\frac{\Gamma(\gamma - i\tilde{\zeta}) \Gamma\left(\frac{1}{2} + i\tilde{\zeta}\right)}{\Gamma\left(\gamma + \frac{1}{2}\right)} F_5 \left(\begin{matrix} \gamma + i\tilde{\zeta}, & \gamma - i\tilde{\zeta} : & 2\gamma + m \\ . & \gamma + \frac{1}{2} : & 2\gamma \end{matrix} \middle| \tau_z, \nu_z \right), \quad (5.35)$$

and therefore the sum in (5.32) takes the form

$$\mathfrak{S} = \frac{(1 - \bar{z})^{-2\gamma}}{\Gamma\left(\gamma + \frac{1}{2}\right) \Gamma(2\gamma) z^m} \left(\frac{z-1}{1-\bar{z}} \right)^m F_5 \left(\begin{matrix} \gamma + i\tilde{\zeta}, & \gamma - i\tilde{\zeta} : & 2\gamma + m \\ . & \gamma + \frac{1}{2} : & 2\gamma \end{matrix} \middle| \tau_z, \nu_z \right). \quad (5.36)$$

Summarizing the above calculations we arrive at the expression of the wave functions:

$$\begin{aligned} \varphi_{z,\gamma,m}(\xi) &= (1 - z\bar{z})^\gamma \sqrt{\frac{2\Gamma(m+2\gamma)}{m!}} i^\gamma (c^{-4})^{i\xi} \Gamma^2(\gamma + i\xi) (\Gamma(i\xi))^{-1} \\ &\quad \times \frac{(1 - \bar{z})^{-2\gamma}}{\Gamma(2\gamma) \Gamma(\gamma + \frac{1}{2})} \left(\frac{z-1}{1-\bar{z}} \right)^m F_5 \left(\begin{matrix} \gamma + i\xi, & \gamma - i\xi : & 2\gamma + m \\ & \gamma + \frac{1}{2} : & 2\gamma \end{matrix} \mid \tau_z, \nu_z \right) \end{aligned} \quad (5.37)$$

as announced in the proposition. Replacing the arguments ν_z and τ_z by their expressions in (5.24) and (5.27), we end the proof. \square

6. RELATIVISTIC BARGMANN-TYPE TRANSFORMS

Now, since we have obtained the expression of the wave functions (5.2), we can apply the coherent states transform formalism [9] to obtain an integral transform $\mathcal{B}_{c,m}^\Pi : L^2(\mathbb{R}_+) \rightarrow \mathcal{A}^{2(\gamma+m)}(\mathbb{D})$ defined by

$$\mathcal{B}_{c,m}^\Pi[f](z) := \left(\mathcal{N}_{2(\gamma+m),m}(z) \right)^{\frac{1}{2}} \langle f, \varphi_{z,\gamma,m} \rangle_{L^2(\mathbb{R}_+)}. \quad (6.1)$$

We precisely state the following precise result.

Theorem 6.1. *The coherent state transform associated with the wave functions (5.2) is the isometry $\mathcal{B}_{c,m}^\Pi : L^2(\mathbb{R}_+) \rightarrow \mathcal{A}^{2(\gamma+m)}(\mathbb{D})$ defined by*

$$\begin{aligned} \mathcal{B}_{c,m}^\Pi[f](z) &:= \frac{(2\gamma-1)^{\frac{1}{2}}}{\sqrt{\pi m!} (1-z)^{2\gamma}} \frac{\sqrt{2\Gamma(m+2\gamma)} (-i)^\gamma}{\Gamma(2\gamma) \Gamma(\gamma + \frac{1}{2})} \left(\frac{\bar{z}-1}{(1-z)(1-z\bar{z})} \right)^m \\ &\quad \times \int_0^{+\infty} \frac{\Gamma^2(\gamma - i\xi) (c^{-4})^{-i\xi}}{\Gamma(-i\xi)} F_5 \left(\begin{matrix} \gamma - i\xi, & \gamma + i\xi : & 2\gamma + m \\ & \gamma + \frac{1}{2} : & 2\gamma \end{matrix} \mid \frac{1-\bar{z}z}{(1-\bar{z})(z-1)}, \frac{1}{1-\bar{z}} \right) f(\xi) d\xi \end{aligned} \quad (6.2)$$

Definition 6.2. *Let $\gamma = (1 + \sqrt{1+2c^4})/2$, then the coherent state transform (6.2) will be called a relativistic Bargmann-type transform attached to the hyperbolic Landau level $\epsilon_m^\gamma := 4m(m+2\gamma-1)$ on the Poincaré disk.*

Corollary 6.3. *For $m=0$, the coherent state transform associated with the wave functions (5.2) is the isometry $\mathcal{B}_{c,0}^\Pi$ mapping the Hilbert $L^2(\mathbb{R}_+)$ onto the Bergman space $\mathcal{A}^{2\gamma}(\mathbb{D})$ of holomorphic functions $\phi : \mathbb{D} \rightarrow \mathbb{C}$ with $\int_{\mathbb{D}} |\phi(z)|^2 (1-z\bar{z})^{2\gamma-2} dv(z) < +\infty$ by*

$$\begin{aligned} \mathcal{B}_{c,0}^\Pi[f](z) &= \left(\frac{2\gamma-1}{\pi} \right)^{\frac{1}{2}} \frac{(-i)^\gamma}{(1-z)^\gamma} \frac{\sqrt{2}}{\Gamma(\gamma + \frac{1}{2}) \sqrt{\Gamma(2\gamma)}} \\ &\quad \times \int_0^{+\infty} (c^{-4})^{-i\xi} \frac{\Gamma^2(\gamma - i\xi)}{(1-z)^{i\xi} \Gamma(-i\xi)} {}_2F_1 \left(\begin{matrix} \gamma - i\xi, \frac{1}{2} - i\xi \\ \gamma + \frac{1}{2} \end{matrix} \mid z \right) f(\xi) d\xi. \end{aligned} \quad (6.3)$$

Proof. We start by putting $m=0$ in the expression (5.2). This gives

$$\begin{aligned} \varphi_{z,\gamma,0}(\xi) &= \frac{\sqrt{2} i^\gamma (c^{-4})^{i\xi}}{\sqrt{\Gamma(2\gamma)} \Gamma(\gamma + \frac{1}{2})} \frac{\Gamma^2(\gamma + i\xi)}{\Gamma(i\xi)} \\ &\quad \times \frac{(1-z\bar{z})^\gamma}{(1-\bar{z})^{2\gamma}} F_5 \left(\begin{matrix} \gamma + i\xi, & \gamma - i\xi : & 2\gamma \\ & \gamma + \frac{1}{2} : & 2\gamma \end{matrix} \mid \frac{1-\bar{z}z}{(1-z)(\bar{z}-1)}, \frac{1}{1-z} \right). \end{aligned} \quad (6.4)$$

We now observe that the two parameters a and a' in the F_5 -sum as denoted in (5.33) are equal in the case of Eq. (6.4). In this situation the reduction ([8, p.136, Eq. (1.7)]):

$$F_5 \left(\begin{matrix} c, & d : & a \\ . & e : & a \end{matrix} \mid \chi, \zeta \right) = {}_2F_1 \left(\begin{matrix} c, d \\ e \end{matrix} \mid \chi + \zeta \right) \quad (6.5)$$

can be applied and enables us to write

$$F_5 \left(\begin{matrix} \gamma + i\zeta, & \gamma - i\zeta : & 2\gamma \\ . & \gamma + \frac{1}{2} : & 2\gamma \end{matrix} \mid \tau_z, \nu_z \right) = {}_2F_1 \left(\begin{matrix} \gamma + i\zeta, \gamma - i\zeta \\ \gamma + \frac{1}{2} \end{matrix} \mid \frac{z}{z-1} \right), \quad (6.6)$$

where we have replaced of the quantities ν_z and τ_z by their above expression respectively. Next, we make use appeal to the Pffaf transformation ([4, p.68]):

$${}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} \mid x \right) = (1-x)^{-a} \cdot {}_2F_1 \left(\begin{matrix} a, c-b \\ c \end{matrix} \mid \frac{x}{x-1} \right) \quad (6.7)$$

to present the Gauss hypergeometric function in right hand side of Eq. (6.6) as

$$(1-\bar{z})^{\gamma+i\zeta} \cdot {}_2F_1 \left(\begin{matrix} \gamma + i\zeta, \frac{1}{2} + i\zeta \\ \gamma + \frac{1}{2} \end{matrix} \mid \bar{z} \right). \quad (6.8)$$

Returning back to (6.4), and inserting (6.8), we arrive at the expression

$$\begin{aligned} \varphi_{z,\gamma,0}(\bar{\zeta}) &= (1-z\bar{z})^\gamma (1-\bar{z})^{-\gamma+i\zeta} i^\gamma \left(c^{-4} \right)^{i\zeta} \frac{\Gamma^2(\gamma+i\zeta)}{\Gamma(i\zeta)} \\ &\times \frac{\sqrt{2}}{\Gamma\left(\gamma+\frac{1}{2}\right) \sqrt{\Gamma(2\gamma)}} \cdot {}_2F_1 \left(\begin{matrix} \gamma + i\zeta, \frac{1}{2} + i\zeta \\ \gamma + \frac{1}{2} \end{matrix} \mid \bar{z} \right). \end{aligned} \quad (6.9)$$

Finally, we write the quantity $(\mathcal{N}_{2\gamma,0}(z))^{\frac{1}{2}} \langle f, \varphi_{z,\gamma,0} \rangle_{L^2(\mathbb{R}_+)}$ for an arbitrary function f in $L^2(\mathbb{R}_+)$. \square

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